Decomposition of graphs excluding induced subgraph

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Terminology

Graphs

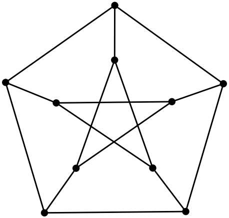

Figure: A graph G

- Vertices or nodes (denoted by $V(G)$)
- Edges (denoted by $E(G)$)

Subgraphs

A subgraph of a graph G is a graph H, where $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

An induced subgraph of a graph G is a subgraph H obtained from G by *deleting* some vertices of G. (*We say that G contains H.*)

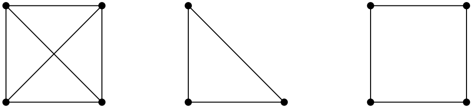

Figure: A graph, an induced subgraph, and a *non*-induced subgraph

Graphs closed under taking induced subgraphs

Hereditary property is a graph property which holds for a graph and is inherited by all its *induced* subgraphs.

Example

A class of graphs that do not contain a clique on 3 vertices is hereditary.

Definition

A class of graphs is hereditary if it is **closed** under taking induced subgraphs. (Closed means that if a graph G is in class $\mathcal{C}$, then for every induced subgraph H of G, the graph H is also in $\mathcal{C}$.)

Hereditary class of graphs

Any hereditary class can be characterized as the class of graphs that **do not contain (or exclude)** any graph in some family $\mathcal{F}$.

- forests $= (C_3, C_4, C_5, C_6, C_7, \dots)$-free
- bipartite graphs $= (C_3, C_5, C_7, \dots)$-free
- chordal graphs $= (C_4, C_5, C_6, C_7, \dots)$-free
- perfect graphs $= (C_3, \overline{C_3}, C_5, \overline{C_5}, C_7, \overline{C_7}, \dots)$-free
- P_4-free graphs, (P_4, C_4)-free graphs, etc...

G is ***F*-free** if no induced subgraph of G is isomorphic to F; and is **$\mathcal{F}$-free** if no induced subgraph of G is isomorphic to each $F \in \mathcal{F}$.

Remark: $\mathcal{F}$ can be a finite/infinite family.

Why forbidding induced subgraphs?

(Possibly) naive answers...

- There are many possibilities to forbid induced subgraphs, so this could lead to publishing many papers.

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Why forbidding induced subgraphs?

In real world, many problems can be formulated as graphs, where:

- vertices represent objects;
- edges represent constraints.

Note:

Removing object is equivalent to removing vertex, which means taking an induced subgraph.

Why forbidding induced subgraphs?

Main concerns:

- How classes of graphs closed under taking induced subgraphs can be described in the most general possible way?
- What properties can be proved about them?
- Are combinatorial problems such as coloring, maximum independent set polynomial-time solvable?

Note:

Many NP-Hard problems (e.g. coloring, maximum independent set) become easy when some configurations are forbidden. (e.g. forests, chordal graphs, perfect graphs).

Graph Decomposition

Decomposition of graph (*in general*)

A decomposition of a *connected* graph G is a set of **edge-disjoint** subgraphs (*not necessarily induced*) of G, say: $H_1, H_2, \ldots, H_n$, such that

$$\bigcup_i H_i = G$$

The edge-disjoint property implies $\forall i, j,\ E(H_i) \cap E(H_j) = \emptyset$.

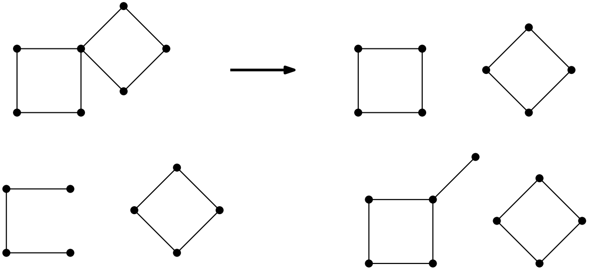


Figure: Decomposition and non-decomposition of a graph

Decomposition of graph

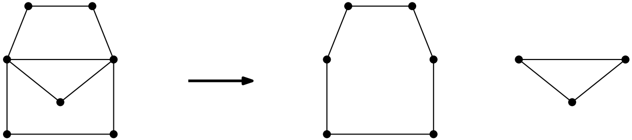


Figure: Cycle decomposition

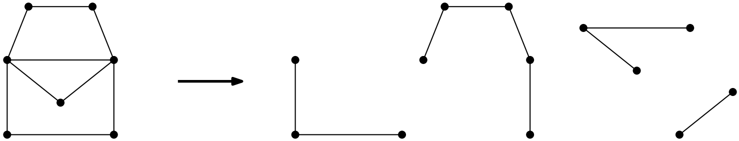


Figure: Path decomposition

Another kind of decomposition

A graph can be decomposed in **many ways** and using **many kinds** of decomposition, depending on how we want to use the decomposition theorem to prove graph properties.

For instance, one can consider that the *edge set* used to decompose the graph is contained in each connected components of the graph.

Example:

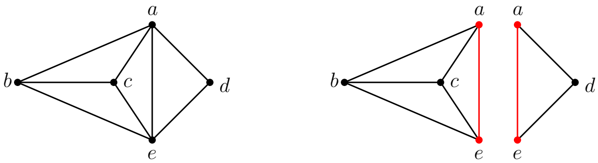
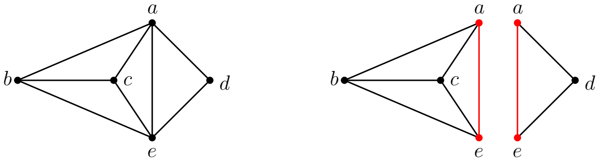


Figure: Decomposition in which the connected components both contains edge $\{a, e\}$



In this case, what we care about is the vertex set that is used to decompose G. Such a vertex set is called cutset of the graph.

Cutset

Let G be a connected graph. A set of vertices $C \subseteq V(G)$ is called a cutset of G if the removal of C from G *disconnects* G, i.e. $V(G) \setminus C$ induces a disconnected graph.

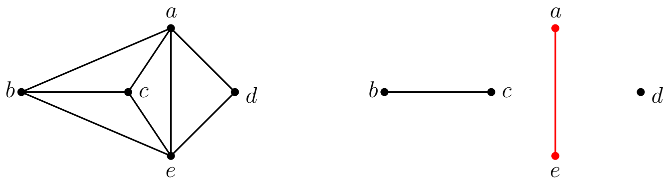


Figure: $\{a, e\}$ is a cutset of graph G that disconnects the graph into two connected components.

Blocks of decomposition

Let G be a graph, C be a cutset in G, and Q be a component of $G \setminus C$. The graph induced by $V(Q) \cup V(C)$ is called a block of decomposition of $G \setminus C$.

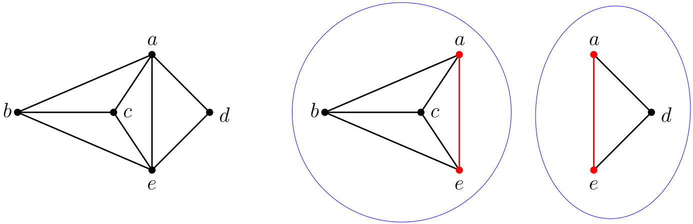


Figure: Blocks of decomposition

Decomposition theorem for a class of graphs

Definition (Decomposition theorem in general...)

A decomposition theorem for a class $\mathcal{C}$ says that every object of $\mathcal{C}$ either:

- belongs to some well-understood *basic* class; or
- it can be broken into smaller pieces according to some well-described rules.

Decomposition theorem for a class of graphs

More formally, for a class of graphs $\mathcal{C}$, we define a set of basic graphs $\mathcal{C}_0$ and a list of graph *decomposition* operations $\mathcal{L}$, s.t. if $G \in \mathcal{C}$:

- either $G \in \mathcal{C}_0$; or
- G can be broken down to smaller graphs G' and G'' using an operation in $\mathcal{L}$ (here we usually use the so-called *cutset*).

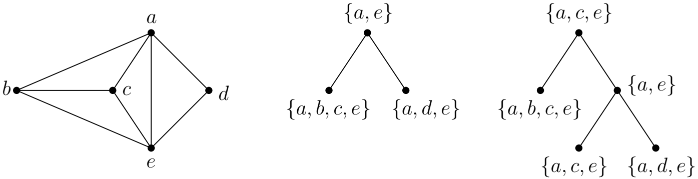
Note:

> *If furthermore, every G can be built from smaller graphs G' and G'' belonging to $\mathcal{C}$ using a compositions operation $\mathcal{L}'$ (the "reverse" operations of $\mathcal{L}$, then it is a structure theorem).*

Example of decomposition

$C \subsetneq V(G)$ is a clique cutset of a connected graph G if C is a cutset of G and C induces a complete graph.

Clique cutset decomposition (Tarjan, 1985)



Example of decomposition theorem

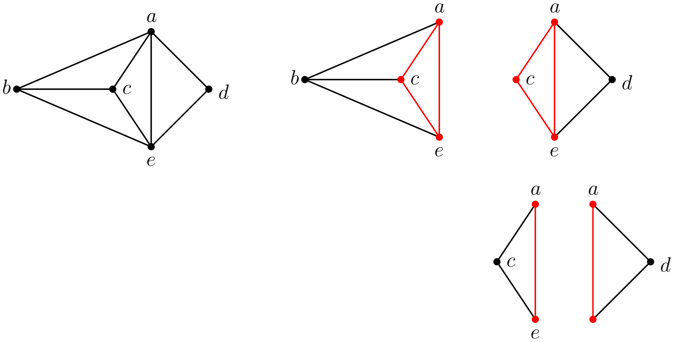


Figure: Decomposition theorem

How decomposition is used for algorithms

The graph-decomposition based algorithm is usually done through the divide-and-conquer approach.

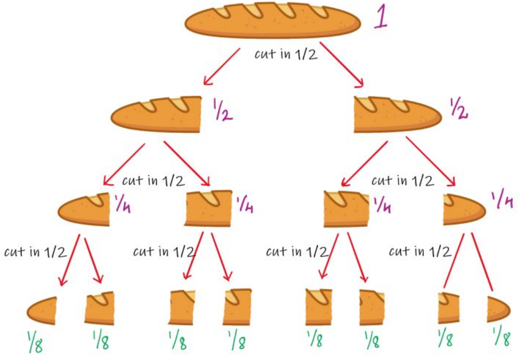


Figure: Illustration of divide-and-conquer algorithm

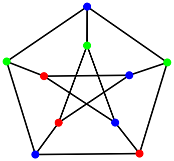
How decomposition is used for *graph recognition*?

Given a class of graphs $\mathcal{C}$. How do we decide if a given input graph G is in $\mathcal{C}$?

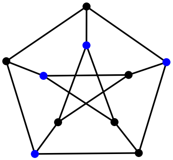
1. Get a decomposition theorem of $\mathcal{C}$ (the decomposition must be *class-preserving*);
2. Decompose G until no decomposition is possible;
3. Check if all graphs obtained from the decomposition are basic graphs of $\mathcal{C}$.

How decomposition is used for *combinatorial problem*

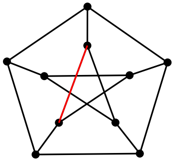
- **Vertex coloring:** assignment of (as minimum possible) colors to the vertices, no adjacent vertices receive the same color
- **Maximum independent set:** finding set of pairwise non-adjacent vertices with maximum cardinality
- **Maximum clique:** finding set of pairwise adjacent vertices with maximum cardinality



coloring
chromatic number : χ



max independent set
independent set number: α



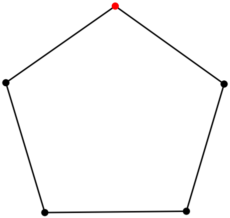
maximum clique
clique number : ω

Part 1

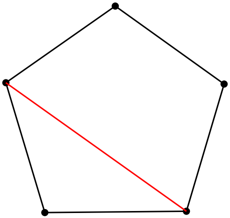
Decomposition theorem of triangle-free even-hole-free (tf-ehf) graphs

Reference: Triangle-Free Graphs Signable without Even Holes [2]

Terminology: what is an *even hole*?


cycle

Terminology: what is an *even hole*?



cycle

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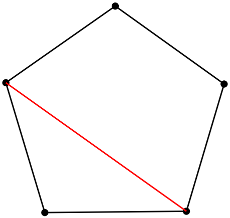
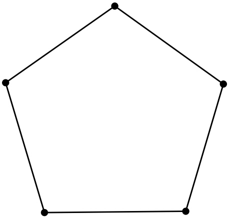
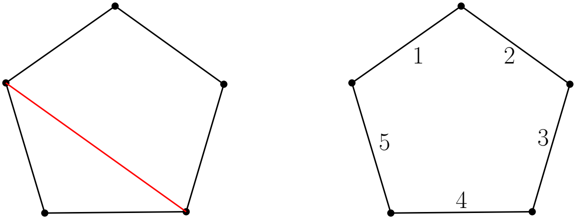


cycle



chordless cycle
(*hole* if it has length ≥ 4)

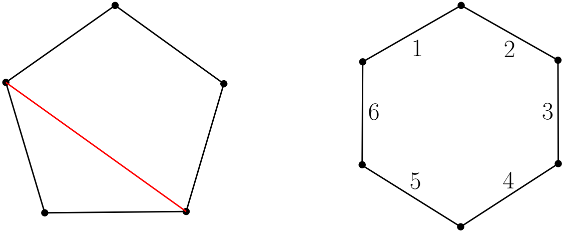
Terminology: what is an *even hole*?



cycle

odd hole

Terminology: what is an *even hole*?

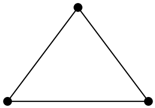


cycle   even hole

A graph is even-hole-free if it does not contain an even hole as an induced subgraph.

What is *triangle*?

Triangle is the complete graph of three vertices.



Triangle-free even-hole-free graphs are graphs that do not contain an *even-hole* and a *triangle* as an induced subgraph.

Forbidden structure in tf-ehf graphs

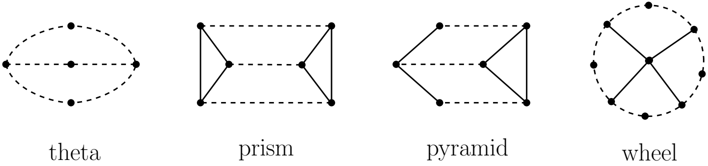


Figure: Truemper configurations; dashed lines represent paths of length at least 1

Every triangle-free even-hole-free graph is (theta, prism, pyramid, even wheel)-free *.

$\mathcal{C}$: the class of (triangle, theta, even wheel)-free graphs

*even wheel is a wheel with an even number of spokes

Decomposition theorem of tf-ehf graphs

Theorem (Decomposition of triangle-free ehf graphs [2)

] *For any graph $G \in \mathcal{C}$* [†]*, one of the following holds.*

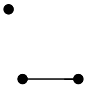
- *G is either a K_1, K_2, a hole, or the cube.*
- *G has a clique cutset.*
- *G contains a wheel, and it can be decomposed with any arbitrarily chosen wheel.*

Assumption: *along the proof, we assume that the studied graph is connected.*

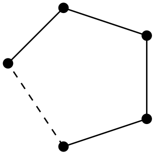
[†] $\mathcal{C}$ is the class of (triangle, theta, even wheel)-free graphs

Sketch of decomposition of tf-ehf graphs

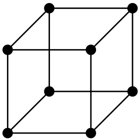
Basic graphs



$K_n,\ n \leq 2$



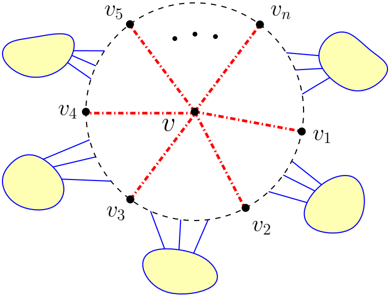
hole



cube

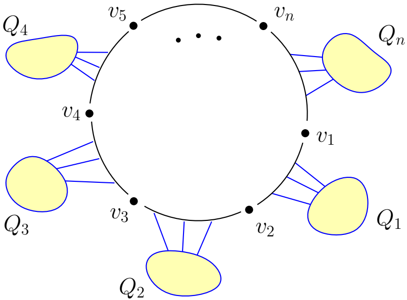
Sketch of decomposition of tf-ehf graphs

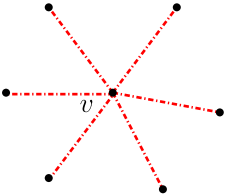
Wheel decomposition



Sketch of decomposition of tf-ehf graphs

Wheel decomposition





Sketch of proof (1): Study clique cutset on the graph

Remark

Let $G \in \mathcal{C}$. Since G is triangle free, the only clique cutset that may exists in G is K_1 or K_2.

Lemma (Clique cutset lemma)

If G has a clique cutset, then G is (theta, prism, pyramid)-free if and only if all blocks of the clique cutset decomposition are (theta, prism, pyramid)-free.

Implication: we may assume that our graph G does not have a clique cutset.

Proof of clique cutset lemma

- Let G be a (theta, prism, pyramid)-free graph, and C be a clique cutset in G.
- For a contradiction, suppose that there exists a block of decomposition Q containing a theta T.
- Since G is theta-free, then T must contains vertices of Q and $G \setminus Q$.
- Study how the vertices of T are positioned in Q and $G \setminus Q$. This will lead to a contradiction.
- Apply a similar analysis, by assuming G contains a prism or a pyramid.

Sketch of proof (2): Study attachment on cube

Lemma

Let $G \in \mathcal{C}$ be a graph containing no K_1 or K_2 cutset. If G contains a cube, then G itself is a cube.

Sketch of proof.

- Assume that G contains cube M.
- Study the attachment of vertices in $G \setminus M$ into M.
  - Study what happens if **all vertices in** $G \setminus M$ **has no neighbor in** M.
  - Study what happens if **a vertex** $x \in G \setminus M$ **has** ≥ 2 **neighbors in** M.
  - Study what happens if **all vertices in** $G \setminus M$ **have at most one neighbor in** M.

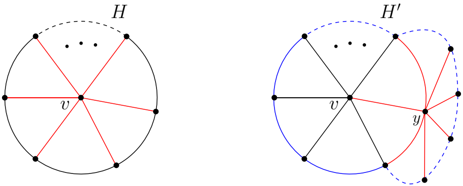
Implication: we may assume that G does not have a clique cutset and the cube (i.e., (K_1, K_2, cube)-free).

Sketch of proof (3): Study attachment on hole

Lemma (Hole attachment lemma)

Let $G \in \mathcal{C}$ that contains no clique cutset, H be a hole in G. Then either:

- *$G = H$;*
- *G contains a wheel (H, v);*
- *G contains a wheel (H', y) as shown in the following figure.*

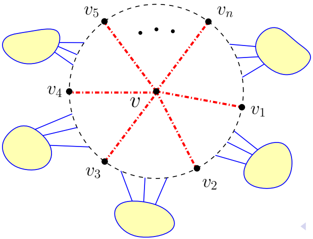


Implication: we may assume that G does not have a clique cutset, and is not a hole and is not a cube.

Sketch of proof (4): Wheel decomposition

Let G be a connected triangle-free graph that contains a wheel (H, v). Let $v_1, \ldots, v_n$ be the neighbors of v in H appearing in this order when traversing H. Then G can be decomposed with a wheel (H, v) if the following holds:

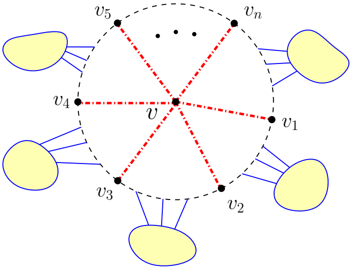
1. $G \setminus \{v, v_1, \ldots, v_n\}$ contains exactly n connected components: $Q_1, \ldots, Q_n$.
2. The intermediate nodes of the sector with endnodes v_i and v_{i+1} belong to Q_i, and no node of Q_i is adjacent to v_j, $j \neq i, i+1$.



Lemma (Wheel decomposition)

Let G be a (triangle, theta, even wheel)-free graph, and G is not a K_1, K_2, a hole, the cube, and G does not contain a clique cutset. Let W be a wheel in G. Then G can be decomposed by W.

Wheel decomposition



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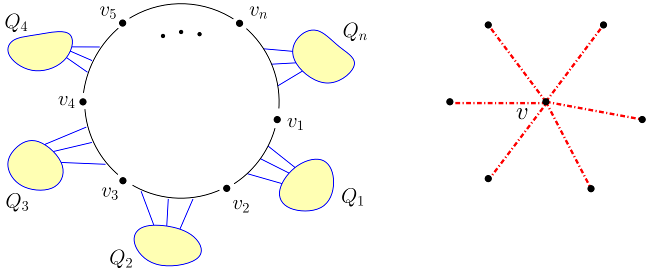
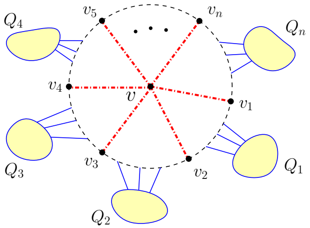
Wheel decomposition



Properties of wheel decomposition

- Let (H, v) be a wheel that decompose G, and $v_i, v_j \in N_H(v)$. Then $\{v, v_i, v_j\}$ contains exactly two connected components.



Wheel decomposition (H, v) where $N_H(v) = \{v_1, \ldots, v_n\}$ contains n connected components.

- Every wheel in G is not "decomposed" (i.e. if W is a wheel in G, then W is contained in a block Q_i of the decomposition).
- If G has no clique-cutset, then every block has no clique cutset.
- If $G \in \mathcal{C}$, then every block $Q_i \in \mathcal{C}$.

Sketch of proof (5): After wheel decomposition

The wheel decomposition can be applied to each block of decomposition which is not a basic graph. This can be done untul in the end, no block of decomposition contains a wheel (i.e. they are all basic graphs).

This follows from the "hole attachment lemma", that says that when each block $Q \in \mathcal{C}$ does not contain any more wheel, then the block of decomposition is a hole.

This proves the "Decomposition Theorem".

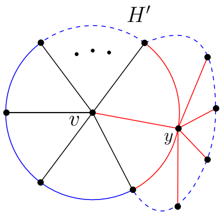
Structure theorem of triangle-free even-hole-free graphs

Reference: Triangle-Free Graphs Signable without Even Holes [2]

Ear

Let H be a hole. A chordless path $P = x \dots z$ is an ear of the hole H if:

- the intermediate nodes of P are in $V(G) \setminus V(H)$;
- x and z have a common neighbor y in H;
- $(V(H) \setminus \{y\}) \cup V(P)$ induces a hole.



Structure theorem of tf-ehf graphs

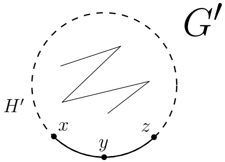
Theorem

Let G be a graph that is triangle-free, not the cube, and containing no K_1 and K_2 cutsets and no cube. Then G is (theta, prism, pyramid)-free if and only if it can be obtained:

- *starting from a hole;*
- *doing a sequence of good ear-addition.*

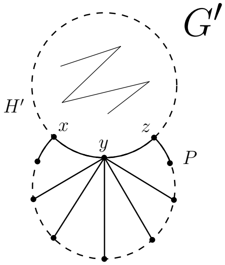
Structure theorem of tf-ehf graphs

Construction: EAR ADDITION



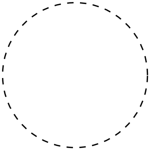
Structure theorem of tf-ehf graphs

Construction: Ear addition



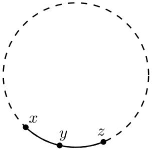
Structure theorem of tf-ehf graphs

Construction: GOOD EAR-ADDITION



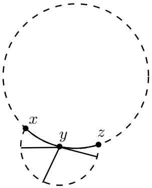
Structure theorem of tf-ehf graphs

Construction: GOOD EAR-ADDITION



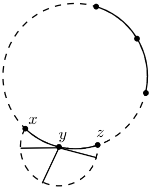
Structure theorem of tf-ehf graphs

Construction: GOOD EAR-ADDITION



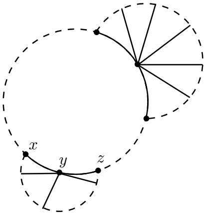
Structure theorem of tf-ehf graphs

Construction: Good ear-addition



Structure theorem of tf-ehf graphs

Construction: GOOD EAR-ADDITION



Structure theorem of tf-ehf graphs

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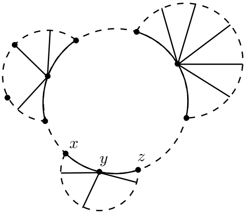

Structure theorem of tf-ehf graphs

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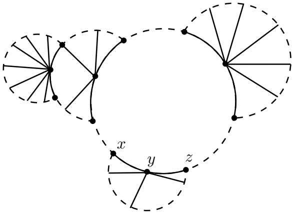

Structure theorem of tf-ehf graphs

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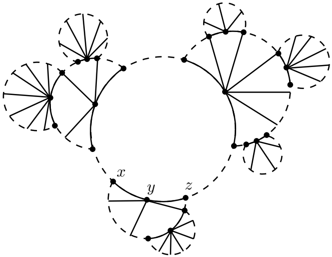
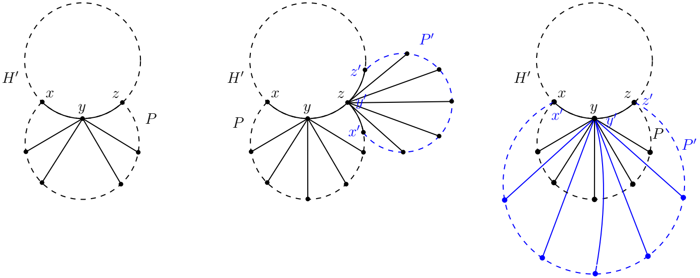


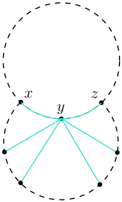
Structure theorem of tf-ehf graphs

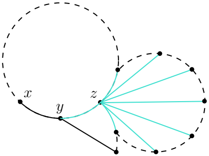
“Bad” ear



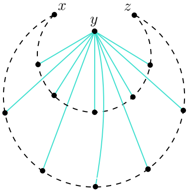
Structure theorem of tf-ehf graphs

"BAD" EAR



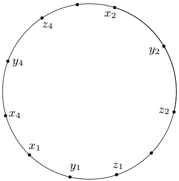


even wheels



Structure theorem of tf-ehf graphs

- Structure theorem of triangle-free case [Conforti, Cornuéjols, Kapoor, Vušković (2000)]



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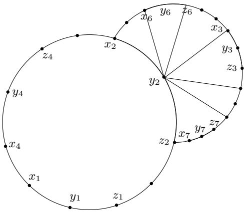

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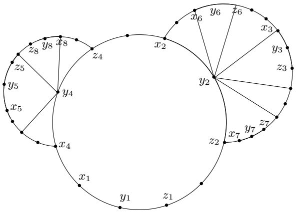

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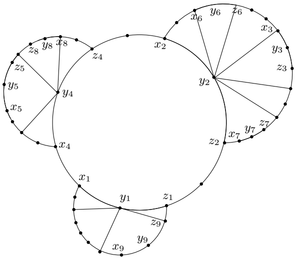

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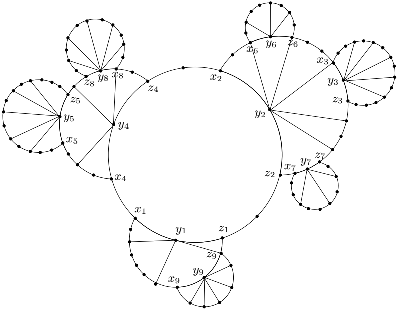


Applications of the decomposition/structure theorems

Reference:

1. Triangle-Free Graphs Signable without Even Holes [2]
2. Structure and algorithms for (cap, even hole)-free graphs [3]

Applications of the decomposition/structure theorems

We will discuss:

1. The use of decomposition theorem for recognizing if a graph is triangle-free even-hole-free.
2. The use of structure theorem for solving combinatorial problems, such as coloring and maximum independent set, through the study of *treewidth*.

Recognizing if a graph in in $\mathcal{C}$

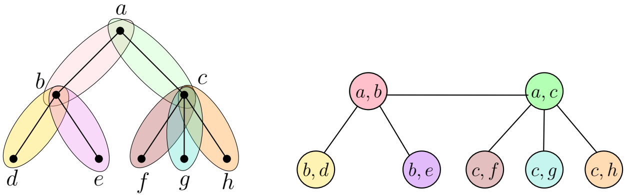
Input: A connected *triangle-free* graph G
Output: YES if G is (theta, prism, pyramid)-free, NO otherwise.

$\mathcal{L}$ is the set of blocks

1. Step 1: If G has a K_1-cutset or K_2-cutset, set $\mathcal{L} = \{G\}$. Otherwise, decompose it with K_1-cutset or K_2-cutset, until it is not decomposable anymore. Let $\mathcal{L}$ be the set of blocks.
2. Step 2: If every graph in $\mathcal{L}$ has one or two nodes, is a hole or a cube, return YES. Otherwise, go to Step 3.
3. Step 3: Study every graph in $\mathcal{L}$ that has more than two nodes, but neither a hole nor a cube. Study if the graph can be decomposed with a wheel. Return NO if a graph is not basic and cannot be decomposed by a wheel.
4. Repeat Step 2, then Step 3 for every graph that is not *basic*.

Treewidth (*intuitively*)

Tree decomposition



- Tree decomposition of G: "gluing" the pieces of subgraphs of G in a tree-like fashion
  - width of T = the size of the largest bag - 1
  - treewidth of G: the minimum over the width of tree decomposition of G

Why treewidth is *algorithmic-ally* powerful?

Theorem (Courcelle, 1990)

Every graph property definable in the monadic second-order logic (MSO) formulas can be decided in linear time on class of graphs of bounded treewidth.

Some graph problems expressible in MSO:

- most combinatorial problems,
- such as: maximum independent set, maximum clique, coloring

Treewidth of tf-ehf graphs

Theorem (3)

Let G be a triangle-free even-hole-free graph, then the treewidth of G is at most 5.

Sketch of proof.

The treewidth of a graph can be computed by applying the following properties.

Lemma

If G is a graph that is contained in a **chordal** *graph H as a subgraph, then the treewidth of G is at most one less than the size of the maximum clique of G*

Sketch of proof (continue).

- By the structure theorem, any graph $G \in \mathcal{C}$ can be formed starting from a hole and sequentially adding *ear-attachment*.
- Then, we can add edges to make the graph chordal.
- It can be proved that the size of the maximum clique of the chordal graph is at most 6.
- Hence the treewidth is at most 5.

Sketch of proof (continue).

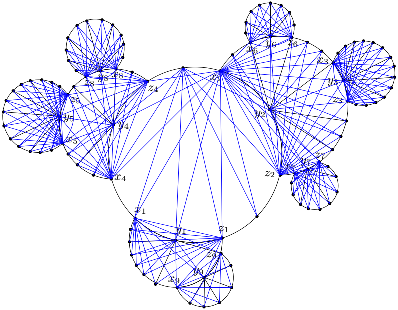


Figure: Sketch of *chordalization* of a graph $G \in cC$

Most combinatorial problems on tf-ehf graphs are polynomial

Implication of treewidth

Since the treewidth is small, then by Courcelle's theorem, many combinatorial problems are poly-time solvable., such as:

- graph coloring; or
- maximum independent set

This is the end of the presentation

References

1. Even-hole-free graphs: a survey (K. Vušković, 2010)
2. Triangle-Free Graphs Signable without Even Holes (M. Conforti, G. Cornu'ejols, A. Kapoor, K. Vušković, 1996)
3. Structure and algorithms for (cap, even hole)-free graphs (K. Cameron, M. V. G. da Silva, S. Huang, K. Vušković, 2016)